The Role of Porosity in Filtration: VI. New Definition of Filtration Resistance

F. M. TILLER
University of Houston, Houston, Texas
MOMPEI SHIRATO
Nagoya University, Nagoya, Japan

The definition of filtration resistance is re-examined in view of the new theory *(12)* of the variation of flow rate with respect to distance through a filter cake. In the new definition it is shown that the filtration resistance depends upon slurry concentration as well as applied pressure. A correction factor modifying the specific filtration resistance previously defined by Ruth *(5, 6)* is developed.

Where the hydraulic pressure variation within a cake is known, methods are presented for calculating the variation of flow rate with respect to distance and the change of filtration resistance in relation to slurry concentration.

The concept of specific filtration resistance α, as contrasted to the permeability constant of D'Arcy's equation, has been used almost exclusively in the development of filtration theory. Ruth (*5*) played a central role in presenting the idea of specific resistance and was largely responsible for the nomenclature used today. The apparent constancy of α during constant pressure filtration was established experimentally by demonstrating that data in the form of volume per unit area v vs. time θ followed a parabolic relation in accord with theoretical prediction based on Ruth's equation. Ruth (*5*) was sufficiently impressed with this functional relationship to propose that the parabolic form of v vs. θ be accepted as an axiom on which mathematical relations for filtration might be derived.

Determination of an average α from v vs. θ data was similar to the calculation of overall heat transfer coefficients on the basis of heat balances in that it did not shed any light on the internal mechanism of the operations. Investigation of local cake conditions as controlling the overall filtration resistance began with the introduction of the permeability-compression cell by Ruth (*6*) who suggested means for relating the average α to the local values α_x. Grace (*2*), Kottwitz and Boylan (*4*), and Shirato and Okamura (*8*) showed reasonable correlation between experimentally determined values of α as compared with calculations based on data obtained with compression-permeability cells.

In 1953 Tiller (*10*) first showed theoretically that the v vs. θ curves for constant pressure filtration were not perfect parabolas. Whenever the pressure drop across the medium was a substantial fraction of the pressure loss across the cake, it was demonstrated that the average filtration resistance was not constant and that the $d\theta/dv$ vs. v curves were not straight. Later (*12*) the two basic assumptions of constant flow rate q independent of distance through the filter bed, and constant average porosity ϵ_{av} (or constant ratio m of mass of wet to mass of dry cake) were found to be invalid. For practical considerations variation in α and m (or ϵ_{av}) can be neglected when the filtration lasts for more than a few minutes.

The newly discovered variation in flow rate throughout the thickness of the bed requires that the concept and definition of filtration resistance be re-evaluated.

FILTRATION RESISTANCE

In Equations (60-61) of paper number I (*10*) and Equations (3-4) of paper number IV (*12*) of this series definitions of the average filtration resistance in accord with past practice were given. In general the filtration resistance α_R of Ruth has been defined as

$$\alpha_R = \frac{p - p_1}{\displaystyle\int_0^{p-p_1} \frac{dp_s}{\alpha_x}} \tag{1}$$

where $g_c p_1 = \mu q_1 R_m$ relates the pressure at the medium p_1 to the flow rate q_1 at the exit from the cake. If p_1 is small, the value of α_R depends only on the total pressure p, and the volume vs. time relation will be parabolic for constant pressure filtration. Frequent lack of good (±20%) agreement and occasional wide variation in calculated and experimental value of α are partially accounted for by weaknesses in the definition given in (1) as well as by experimental difficulties.

In order to develop a better definition of α (*7*), begin with the basic flow equation in the form

$$g_c \frac{dp_s}{dw_x} = -\mu \alpha_x q_x \tag{2}$$

where the subscripted q_x is employed to emphasize that the rate of flow varies with the distance.* Actually q_x is a function of both x and θ and should be written as $q(x,\theta)$. Since the local α_x is assumed to be a function only of $p_s = p - p_x$, Equation (2) is placed in the form (limits from Figure 1)

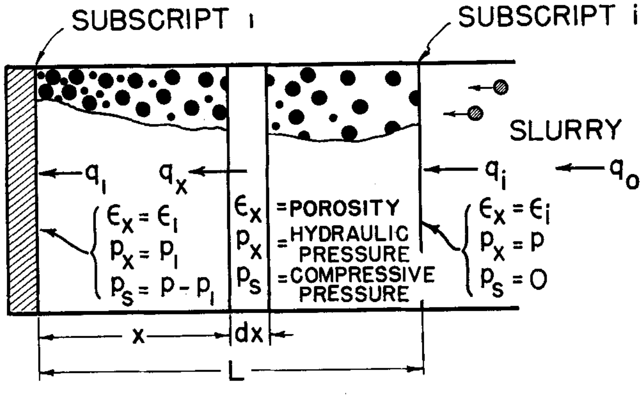

Fig. 1. Schematic diagram of cake.

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[*] In this paper x and w_x are taken in opposite directions to those used conventionally and previously in this series. A minus sign must be placed in the equation to account for the change. If comparisons are made with Equations (3) and (4) of a previous article (*11*) written by the author, there may be confusion as the previous Equation (3) appeared erroneously without a minus sign. Development in this paper and a subsequent one being written by the authors are considerably simplified by such a change. After careful consideration is was deemed essential to measure distance from the fixed medium rather than the moving cake surface. The sign in Equation (18) is also changed.

$$\int_0^{p-p_1}\frac{dp_s}{\alpha_x} = -\frac{\mu}{g_c}\int_w^0 q_x dw_x = \frac{\mu}{g_c}\int_0^w q_x dw_x \qquad (3)$$

in preparation for integration. Customarily in the definition of α in the past q_x has been considered constant although it may vary twofold or more.

Rather than assume $q_x = q_1$ and factor it out from under the integral sign, q_x will be assumed to vary with x or w_x. The integral on the right-hand side of (3) may be multiplied and divided by $q_1 w$ and placed in the form

$$q_1 w\int_0^w \frac{q_x dw_x}{q_1 w} = q_1 w\int_0^1\left(\frac{q_x}{q_1}\right)d\left(\frac{w_x}{w}\right) = Jq_1 w \qquad (4)$$

where J is equal to the integral with limits from zero to one. With this change both sides of Equation (3) may be divided into $p - p_1$ and rearranged in a manner which serves to define the new filtration resistance α_T (7) as follows:

$$\alpha_T = \left[\int_0^1\left(\frac{q_x}{q_1}\right)d\left(\frac{w_x}{w}\right)\right]\frac{(p-p_1)}{\displaystyle\int_0^{p-p_1}\frac{dp_s}{\alpha_x}} = \frac{g_c(p-p_1)}{\mu q_1 w} \qquad (5)$$

The new definition of specific resistance is related to the older definition by

$$\alpha_T = J\alpha_R \qquad (6)$$

By placing $q_1 = dv/d\theta$ and $g_c p_1 = \mu q_1 R_m$ in Equation (5) it is possible to rearrange (5) into the well-known form

$$q_1 = \frac{dv}{d\theta} = \frac{g_c p}{\mu(J\alpha_R w + R_m)} \qquad (7)$$

The conventional equation previously employed results when J is equated to unity. It will be shown later that J approaches unity for dilute slurries which have represented the basis for most data reported in the literature. The correction factor $J = q_{av}/q_1$ where q_{av} is based upon integration with respect to w_x; it will be less than unity as $q_{av} < q_1$.

Previously (*12*) it was shown that the minimum flow rate q_i in the infinitesimal surface layer could be related to q_1 by*

$$\frac{q_i}{q_1} = 1 - \left(\frac{\epsilon_i - \epsilon_{av}}{\epsilon_{av}}\right)\left(\frac{m-1}{1-ms}\right)s$$
$$= 1 - (m_i - m)(1-\epsilon_i)\frac{s}{1-ms} \qquad (8)$$

When s is at its maximum and equals to $1/m_i$, the ratio q_i/q_1 will be ϵ_i. Since q_i is the smallest value of q_x, limits may be placed on J

$$\epsilon_i \leqq q_i/q_1 \leqq J \leqq 1 \qquad (9)$$

where q_i/q_1 can be calculated from Equation (8). Accurate calculation of q_i suffers from a lack of availability of experimental values of ϵ_i. Frequently data from permeability compression cells are not available at pressures below 0.5 lb./sq. in., and unfortunately there may be wide changes in porosity at lower pressures (Figure 8). Uncertain extrapolation must be used to obtain ϵ_i even for moderately compressible materials such as kaolin (*9*).

Shirato and Okamura (*8*) in 1956 presented data indicating that experimental values of the filtration resistance obtained for thick slurries were smaller than values obtained with dilute slurries. Calculations employing compression permeability data in Equation (1), on the assumption that q_x was constant throughout the cake, gave values which were higher than the experimentally determined resistances for thick slurries. Although Shirato and Okamura were criticized after presenting their findings, the present paper indicates the basic theoretical validity of their results. In general calculation of the value of J in Equation (4) requires solution of the partial differential equation presented in paper number IV (*12*) of this series. It would be expected that J would decrease with increasing slurry concentration and would change with pressure.

Variation of ϵ_x, p_x/p, and q_x/q_1 with x/L

In the new definition of filtration resistance presented in Equation (5) it is important to know how $\int_0^w q_x dw_x$ varies during the course of a filtration. Under the most general conditions it is necessary to solve the partial differential equation presented in Part IV (*12*) of this series in order to determine the relation of the integral to the filtration variables. However under very wide circumstances conclusions may be drawn from which important information can be garnered concerning $\int_0^w q_x dw_x$. Both experimental work and filtration theory of dilute slurries point to the quantities q_x/q_1, p_x/p, and ϵ_x as being functions of x/L which are essentially independent of time. Under such circumstances it will be shown that α_T is constant during a constant-pressure filtration just as is true for α_R. As previously demonstrated for α_R (*10, 12*) α_T would not be constant when the medium offers a substantial fraction of the total resistance.

In determining ϵ_x as a function of cake thickness it is difficult to make direct measurements. However indirect determinations can be made by finding the variation of the hydraulic pressure with time at a fixed point in the bed. When one has the value of hydraulic pressure p_x the solid compressive pressure p_s can be obtained from the expression $p_s = p - p_x$. If equilibrium conditions have been reached in the cake, the porosity is assumed to be a function of p_s alone and equal to the same value as found in a compression-permeability cell at the same pressure.

Shirato (*7*) developed a method for finding the hydraulic pressure variation using specially designed manometers connected to probes within a cake. In Figure 2 the types of data which were obtained are illustrated for a Hong Kong pink Kaolin slurry having a mass fraction of solid equal to 0.222 and filtered at a constant pressure of 42.7 lb./sq. in. As long as a probe remained outside of the

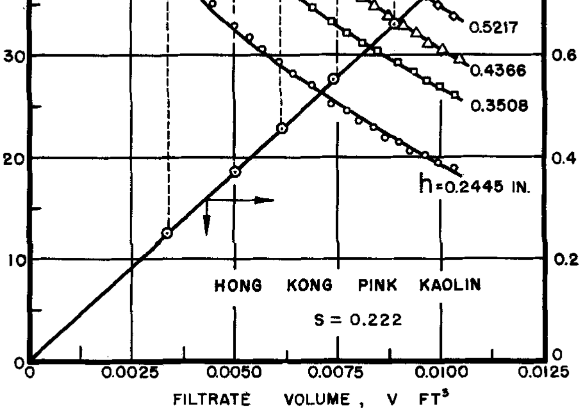

Fig. 2. Hydraulic pressure and cake thickness vs. filtration volume.

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* The derivation previously (*12*) used for finding the ratio of the flow rates was in error. The reported value $q_i/q_1 = (1 - m_i s)/(1 - ms)$ should be replaced by Equation (11). See the end of this article for correct derivation.

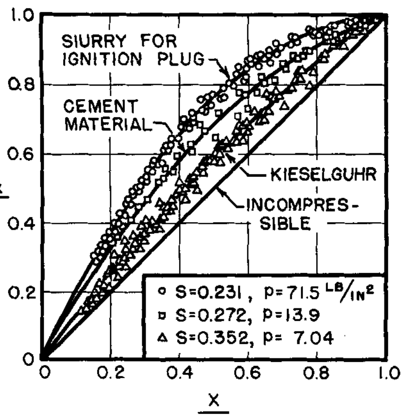

Fig. 3. Hydraulic pressure vs x/L.

cake, the pressure was constant and equal to the applied filtration pressure. As soon as the cake increased in thickness sufficiently to envelope a probe, the pressure began to fall. The thickness of the cake could be determined accurately as a function of time or volume filtered by observing the time at which the pressure on a given probe began to fall.

Based upon experiments similar to the one illustrated in Figure 2, plots of p_x/p vs. x/L were prepared as shown in Figure 3. It is apparent that time does not appreciably affect the relationship. Previously in paper number IV of this series (*12*) it was demonstrated with theoretical calculations (Figure 6 of reference *12*) that ϵ_x was a function of x/L alone and not θ for a dilute slurry of kaolin after 1 min. of filtration. In the first few seconds calculated values for the filtration of kaolin indicated that porosity was a function of time as well as the normalized distance x/L. However a limiting curve of ϵ_x as a function of x/L was rapidly approached. When the average experimental filtration resistance becomes constant and the volume vs. time discharge curve is parabolic in a constant pressure filtration, it would be expected that ϵ_x would have reached its limiting relation as a unique function of x/L. In short filtrations with thick slurries, where α is expected to vary with time, the assumption of a unique relation between ϵ_x and x/L would not be expected to be valid. In addition to the previous evidence empirical expressions relating ϵ_x to the normalized distance x/L were developed in paper V (*13*) of this series. Consequently it is felt that under a wide variety of conditions it may be assumed that ϵ_x depends only on x/L.

Since the solid is distributed nonuniformly throughout the cake, it is necessary to show that w_x/w is a unique function of x/L. The differential dw_x can be written as

$$dw_x = \rho_s(1 - \epsilon_x)\,dx \tag{10}$$

and the total mass of cake w is given by

$$w = \rho_s\,(1 - \epsilon_{av})\,L \tag{11}$$

Dividing (10) by (11) one gets an equation in which the normalized variables can be presented as

$$\int_o^{w_x/w} d\left(\frac{w_x}{w}\right) = \frac{w_x}{w} = \int_o^{x/L} \frac{1 - \epsilon_x}{1 - \epsilon_{av}} d\,(x/L) \tag{12}$$

Since ϵ_x is assumed to be a function of x/L and ϵ_{av} is constant, w_x/w is a function of x/L alone.

In order to prove that q_x/q_1 is a function of x/L, a basic equation relating flow rate to distance will be derived. The void volume per unit cross-sectional area in a differential section of cake equals ϵdx, and the total void volume, as illustrated in Figure 5, is given by

$$\frac{\text{void volume}}{\text{unit area}} = \epsilon_{av}\,L = \int_o^L \epsilon_x dx \tag{13}$$

If the upper limit in (13) is replaced by x, the integral represents the void volume per unit area contained in distance x. Since the porosity is decreasing throughout the cake contained in distance 0 to x, the rate of flow into this portion of the cake q_x minus the rate of flow out q_1 at the medium (see Figure 1) must equal to the rate of gain of liquid. Mathematically a material balance gives

$$q_x - q_1 = \frac{\partial}{\partial\theta}\int_o^x \epsilon_x dx = \int_o^x \frac{\partial\epsilon_x}{\partial\theta}dx \tag{14}$$

where it is assumed that ϵ_x is a function which can be differentiated under the integral sign. If Equation (14) is differentiated with respect to x, there results

$$\frac{\partial q_x}{\partial x} = \frac{\partial\epsilon_x}{\partial\theta} \tag{15}$$

which was previously developed (*12*) with an altered sign. This equation was originally derived by Terzaghi (*9*) in 1923 and is basic to rate-of-consolidation theory in soil mechanics.

If it is assumed that $\epsilon_x = f(x/L)$ and that L is a function of θ, then (15) may be modified as follows with assistance of the chain rule of differentiation:

$$\frac{\partial q_x}{\partial x} = \frac{d\epsilon_x}{d(x/L)}\,\frac{d(x/L)}{dL}\,\frac{dL}{d\theta} = -\frac{d\epsilon_x}{d(x/L)}\,\frac{x}{L^2}\,\frac{dL}{d\theta} \tag{16}$$

It can be inferred from (16) that at the medium where $x = 0$ the derivative $\partial q_x/\partial x = 0$. Substituting (16) in (14) one obtains

$$q_1 - q_x = \frac{dL}{d\theta}\int_o^x \frac{d\epsilon_x}{d(x/L)}\,\frac{x}{L^2}dx \tag{17}$$

Cancelling $d(x/L)$ with dx/L one gets

$$q_1 - q_x = \frac{dL}{d\theta}\int_{\epsilon_1}^{\epsilon_x} (x/L)d\epsilon_x \tag{18}$$

where ϵ_1 and ϵ_x are limits corresponding to o and x with the change in variable of integration from x to ϵ_x. Carrying out the integration over the entire cake and dividing the result into (18) one obtains

$$\frac{q_1 - q_x}{q_1 - q_i} = \frac{1 - q_x/q_1}{1 - q_i/q_1} = \frac{\int_{\epsilon_1}^{\epsilon_x} (x/L)\,d\epsilon_x}{\int_{\epsilon_1}^{\epsilon_i} (x/L)\,d\epsilon_x} \tag{19}$$

The ratio of q_i to q_1 is independent of time and can be obtained from (8). Substitu[illegible] for q_i/q_1 and solving for q_x/q_1 one gets

$$\frac{q_x}{q_1} = 1 - \frac{(\epsilon_i - \epsilon_a \quad 1)s}{\epsilon_{av}(1 - ms)}\,\frac{\int_{\epsilon_1}^{\epsilon_x} (x/L)\,d\epsilon_x}{\int_{\epsilon_1}^{\epsilon_i} (x/L)\,d\epsilon_x} \tag{20}$$

This expression can be modified by making use of the identity

$$\int_{\epsilon_1}^{\epsilon_x} (x/L)\,d\epsilon_x = \epsilon_x(x/L) - \int_o^{x/L} \epsilon_x\,d(x/L) \tag{21}$$

If ϵ_{av} represents the average porosity for the entire cake and ϵ_{avx} represents the average for the portion of the cake between the septum and the normalized distance x/L from the septum, then (21) can be written

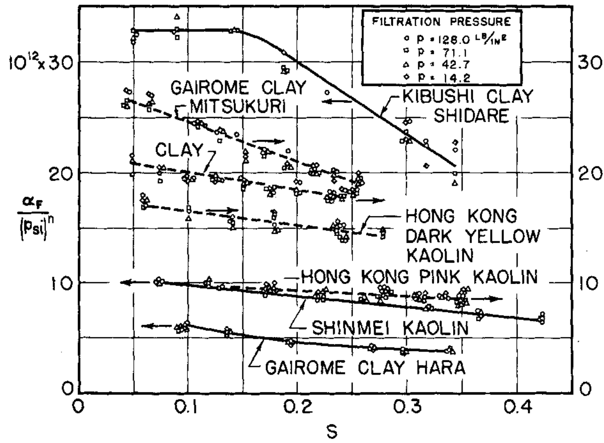

Fig. 4. $\alpha_F/(\text{lb.sq.in.})^n$ vs. slurry concentration.

$$\int_{\epsilon_1}^{\epsilon_x} (x/L)\, d\epsilon_x = (\epsilon_x - \epsilon_{avx})\,(x/L) \tag{22}$$

Substitution of (22) into (20) yields

$$\frac{q_x}{q_1} = 1 - \frac{(\epsilon_x - \epsilon_{avx})\,(m-1)}{\epsilon_{av}(1-ms)}\, s\, \frac{x}{L} \tag{23}$$

Substitution for m in terms of porosity gives

$$\frac{q_x}{q_1} = 1 - \frac{\rho(\epsilon_x - \epsilon_{avx})\; s\;(x/L)}{\rho_s(1-s)\;(1-\epsilon_{av}) - \rho s \epsilon_{av}} \tag{24}$$

The value of ϵ_{av} depends chiefly on the applied filtration pressure when the medium resistance is negligible. Therefore q_x/q_1 is a function x/L alone at constant s and p.

Returning to Equation (4) one can see that J will be a function of p and s. Consequently the specific resistance will be constant during a constant pressure filtration. Substituting (23) in the equation defining J and integrating one gets

$$J = \int_0^1 \left(\frac{q_x}{q_1}\right) d\left(\frac{w_x}{w}\right)$$

$$= 1 - \frac{(m-1)\,s}{\epsilon_{av}(1-ms)} \int_0^1 (\epsilon_x - \epsilon_{avx}) \left(\frac{x}{L}\right) d\left(\frac{w_x}{w}\right) \tag{25}$$

The essence of this paper is contained in Equation (25). Basically it states that the conventional filtration resistance must be multiplied by a factor which depends upon porosity variation and slurry concentration. While the influence of porosity is not unexpected, it is surprising to find that the slurry concentration may exert a pronounced effect on cake resistance. When s approaches zero, J approaches unity, indicating that filtr[illegible]ion resistance for dilute slurries would be in accord [illegible]h previous [illegible]eory. On the other hand as s becomes [illegible] the v[illegible] of J decreases but always remains large [illegible] ϵ_i. It [illegible]uld be remembered that the assumptio[illegible] is [illegible] function of x/L alone and all subsequent eq[illegible]tions depend upon the medium resistance being negligible.

In the last section of this paper a method for calculating J will be presented for conditions under which the porosity distribution is known. In a future paper, more general procedures for determining J will be developed.

The general decrease of filtration resistance with increasing slurry concentration can be seen qualitatively in Figure 4 for a number of clays investigated by Shirato (7). The data are plotted in the form of α_F/p^n vs. s, where α_F represents an experimentally determined value which should be closer to α_T than α_R. Empirically the resistance data can be represented as functions of p and s as follows, where Ψ represents the pressure in lbs./sq. in.

Gairome clay, Hara

$$\alpha_F = 2.50\,(10^{12})\,s^{-0.38}\Psi^{0.60} \qquad s > 0.10 \tag{26}$$

Kibushi clay

$$\alpha_F = 3.28\,(10^{13})\Psi^{0.42} \qquad 0.15 > s > 0.05 \tag{27}$$

$$\alpha_F = 4.31\,(10^{13})\,(1 - 1.52s)\Psi^{0.42} \qquad s > 0.15 \tag{28}$$

For the other materials the data may be represented by

$$\alpha_F = A_o\,(1 - Ds)\Psi^N \qquad s > 0.05 \tag{29}$$

where the constants A_o, D, and N have the values

	A_o	D	N
Hong Kong pink kaolin	1.01 (10^{12})	0.478	0.332
Hong Kong dark yellow kaolin	1.78 (10^{12})	0.741	0.465
Shinmei kaolin	1.08 (10^{13})	0.905	0.40
Gairome clay, Mitsukuri	2.82 (10^{12})	1.272	0.609
Clay (mixture)	2.16 (10^{12})	0.744	0.471

The data shown in Figure 4 and represented by the empirical Equations (26) to (29) generally follow the trend predicted by Equation (25). Insufficient data were available for making a precise numerical comparison of theory and experiment.

Since the filtration resistance decreases substantially with increasing slurry concentration, commercial operation at high concentrations would require smaller filter areas than would be predicted on the basis of experimental work with thin slurries. A filter operating with high values of s would have a marked advantage for materials of the type represented by Equations (26) to (29). Where flocculation occurs or cake structure changes with increasing slurry concentration, the values of α_x may be further affected, and the effect of increasing s on α_T will be greater than predicted by Equation (25) alone.

PHYSICAL INTERPRETATION OF EQUATION

In Figure 5 a plot of ϵ_x vs. x is shown in relation to the integral $\int x d\epsilon_x$ as it occurs in previous equations. If there were no squeezing and compressive action within the cake, there would be a uniform porosity equal to ϵ_i, the value in an infinitesimal surface layer. The area ABDO in the figure, equal to $\epsilon_i L$, is proportional to the volume of liquid per square foot of cross-sectional area which would

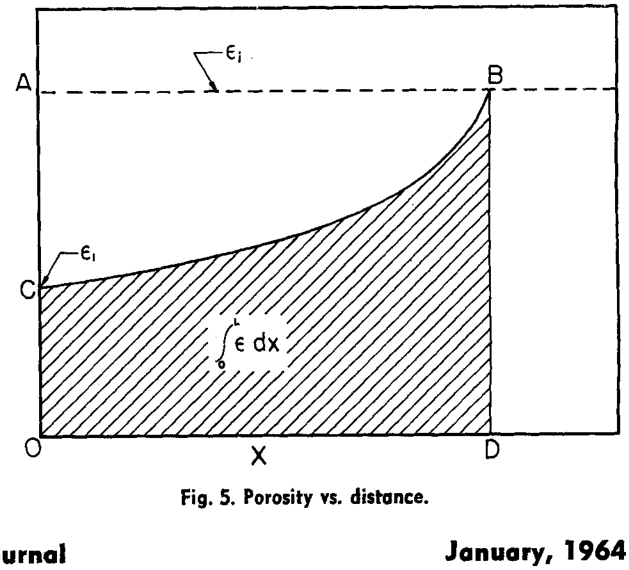

Fig. 5. Porosity vs. distance.

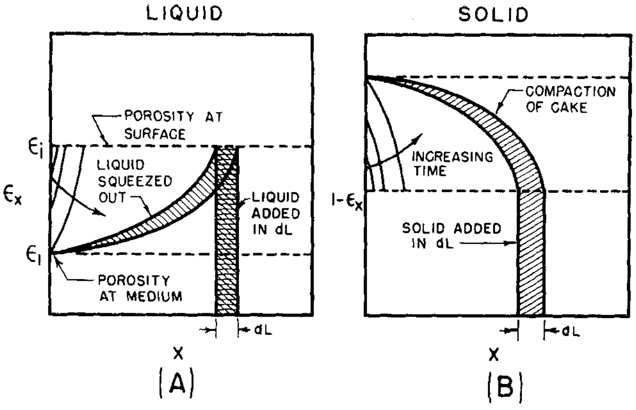

Fig. 6. Porosity vs. distance, increasing cake thickness.

remain in the cake if these were no compression. The area CBDO, which equals $\int_0^L \epsilon_x dx$, represents the volume of liquid retained in the interstices of the cake, and consequently the area ABC equaling $\int_{\epsilon_1}^{\epsilon_i} x d\epsilon_x$ is proportional to the liquid which was first deposited in the cake and then was squeezed out as the cake was compressed during filtration.

In Figure 6 the increase in thickness of the cake dL during time $d\theta$ is illustrated. In Figure 6A the liquid squeezed from the cake is represented by the singly cross-hatched area. The doubly cross-hatched area represents the liquid added to the cake and is always larger than the liquid removed. In Figure 6B $(1 - \epsilon_x)$ is plotted against x. The integral $\int_0^L (1-\epsilon_x)\, dx$ corresponds to the volume of solid material and gives the compaction effect in the cake.

In Figure 7 the term $x(\epsilon_x - \epsilon_{avx})$ appearing in Equation (22) is illustrated as the rectangle CDEF which equals the area CDG or $\int x d\epsilon_x$ between ϵ_1 and ϵ_x. The quantity $(\epsilon_i - \epsilon_{av})L$ corresponds to the total area ABDGEC lying to the left of the porosity curve. The ratio of $(q_x - q_i)/(q_1 - q_i)$ is given by the ratio of the cross-hatched area ABDC to the total area ABDGEC. If ϵ_x vs. x/L data are available, it is possible to calculate q_x/q_1 with the use of the equations which have been developed. In the next section calculations of q_x/q_1 and J will be illustrated.

ANALYSIS OF EXPERIMENTAL DATA

If α_x and ϵ_x are available from compression-permeability experiments and p_x vs. x/L data have been obtained, it is possible to construct the q_x vs. x/L curve. Shirato (7) has obtained such data for a number of substances as illustrated in Figure 3, where p_x/p is plotted against x/L for ignition plug slurry, cement material, and kiselguhr. In Figure 8 graphs of ϵ_x vs. p_s as obtained for compression-permeability cell data are shown for ignition plug and cement material. When one makes use of the basic assumption in filtration that the porosity in a filter cake is the same as the porosity in a compression permeability cell when $(p - p_x)$ in the cake equals p_s in the cell, it is possible to combine the data of Figures 3 and 8 to produce the plot of ϵ_x vs. x/L for cement material in Figure 9A.

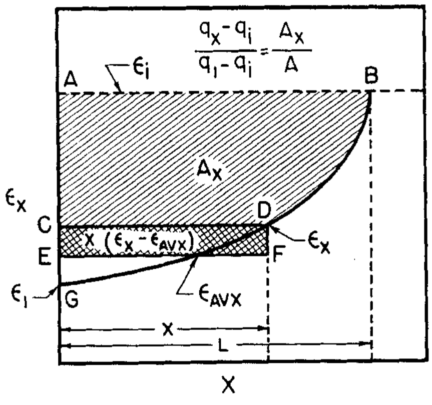

Fig. 7. Area representing liquid squeezed from cake.

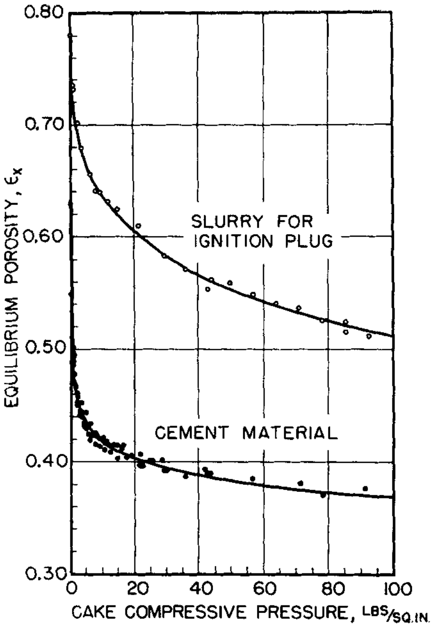

Fig. 8. Equilibrium porosity vs. compressive pressure.

In order to obtain a plot of q_x/q_1 vs. x/L or w_x/w it is necessary to have an accurate value of ϵ_i or m_i. However reference to Figure 8 shows that the large slope of the ϵ_x vs. p_s curve near $p_s = 0$ makes accurate determination of ϵ_x quite difficult. For the cement material the value of ϵ_i will lie somewhere between 0.55 and 0.63. The nature

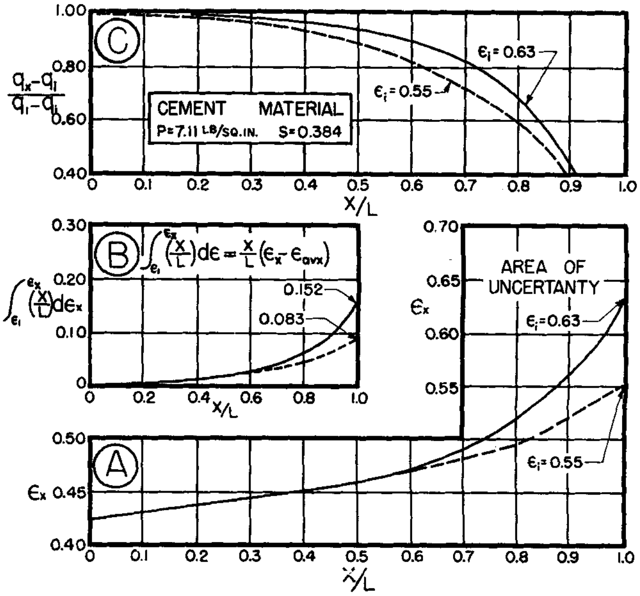

Fig. 9. Numerical integration to obtain flow variation.

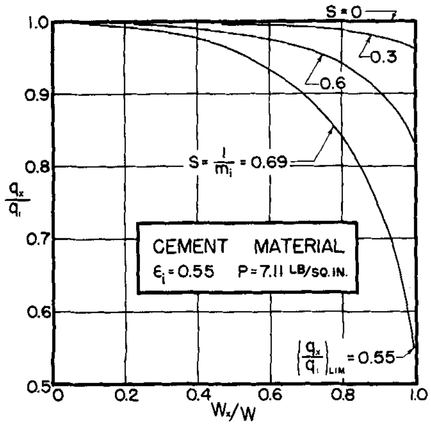

Fig. 10. q_x/q_1 vs. w_x/w.

of the difficulty is illustrated in Figure 9B where the integral $\int(x/L)d\epsilon_x$ is shown for cement material. The area to the left of the ϵ_x vs. x/L curve marked "area of uncertainty" in Figure 9A is a large fraction of the total area. In Figure 9B the total value of the integral is shown as approaching 0.083 or 0.152 depending upon whether ϵ_i is chosen as 0.55 or 0.63. The results of using the two limiting values of the surface porosity ϵ_i are illustrated in Figure 9C where $(q_x - q_i)/(q_1 - q_i)$ is plotted against x/L. The differences are not large, and the effect on calculating α would be small in this case.

Plots of q_x/q_1 for various values of slurry concentration of cement material are shown in Figure 10 as a function of w_x/w. In order to change from x/L to w_x/w Equation (12) can be employed. Combined with (23) Equation (12) yields values of q_x/q_1 and w_x/w as functions of x/L and s. In making calculations for Figure 10 it was assumed as an approximation that m was independent of s. The limiting curve labeled $(q_x/q_1)_{\text{LIM}}$ is identical with the curve plotted in Figure 9C and marked $\epsilon_i = 0.63$. In the limiting condition the slurry is increased in concentration until it is a soupy solid equivalent in porosity to the surface layer of cake with $1 - m_i s = 0$ or $s = 1/m_i$. The flow ratio q_i/q_1 at the surface would equal ϵ_i as calculated by Equation (8). With $q_i/q_1 = \epsilon_i$, the quantity $(q_x - q_i)/(q_1 - q_i)$ simply reduces to $(q_x/q_1)_{\text{LIM}}$.

The area under the $(q_x/q_1)_{\text{LIM}}$ vs. w_x/w curve is 0.908 which is the minimum value of J. As s decreases, the area under the q_x/q_1 curves approaches unity. Where flocculation occurs with changing s and the integral $\int dp_s/\alpha_x$ is affected, the value of α_R as defined in Equation (1) would become a function of s. Under these circumstances the effect of s on $\alpha_T = J\alpha_R$ would be due to changes in both J and α_R.

CONCLUSION

It has been demonstrated that the definition of specific filtration resistance as previously employed in the literature must be modified. The newly defined resistance gives theoretical values which decrease with increasing slurry concentration in accord with the trends of experimental data.

With the hydraulic pressure distribution within a bed used, porosity and flow rate were calculated as a function of normalized distance x/L.

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NOTATION

A_o, D = constants defined in Equation (29), dimensions meaningless
g_c = conversion factor, poundal/pound force, (lb. mass) (ft.)/(lb. force) (sec.^{2})
J = ratio of α_T/α_R, dimensionless
L = cake thickness, ft.
m = ratio of mass of wet to mass of dry cake, dimensionless
m_i = value of m in infinitesimal surface layer of cake
N = constant defined in Equation (29), dimensions meaningless
p = applied filtration pressure, lb. force/sq. ft.
p_s = solid compressive pressure at distance x from medium, lb. force/sq. ft.
p_x = hydraulic pressure at distance x from medium, lb. force/sq. ft.
p_1 = pressure at interface of medium and cake, lb. force/sq. ft.
q_x = rate of flow of liquid in cake at distance x from medium, cu. ft./(sq. ft.) (sec.)
q_1 = value of q at interface of medium and cake where $x = 0$
q_i = value of q in infinitesimal surface layer of cake where $x = L$
R_m = medium resistance, 1/ft.
s = fraction solids in slurry, dimensionless
x = distance from medium, ft.*
w = total mass of dry solids per unit area, lb. mass/sq. ft.
w_x = mass of solids per unit area in distance x from surface of cake, lb. mass/sq. ft.

Greek Letters

α_F = filtration resistance determined by experiment, ft./lb. mass
α_R = filtration resistance previously used in literature as defined by Ruth (5, 6), see Equation (1), ft./lb. mass
α_T = newly defined filtration resistance, see Equation (5), ft./lb. mass
α_x = local value of filtration resistance at pressure p_s, ft./lb mass
ϵ_i = porosity in infinitesimal surface layer of cake, dimensionless
ϵ_x = local porosity at distance x from medium, dimensionless
ϵ_{av} = average porosity of entire cake, dimensionless
ϵ_{avx} = average porosity over first x feet of cake, dimensionless
θ = time, sec.
μ = viscosity, lb. mass/(ft.) (sec.)
ρ = density of liquid, lb. mass/cu. ft.
ρ_s = true density of solids, lb. mass/cu. ft.
Ψ = pressure, lb. force/sq. in.

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* Previously the author (*10, 11, 12, 13*) used x as the distance from the surface of the cake.

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ERRATUM

Professor S. Okamura of Nagoya University has pointed out an error in Equations (12) and (13) in the article "The Role of Porosity in Filtration: IV, Constant Pressure Filtration," [*A.I.Ch.E. Journal*, **6**, 595 (1960)]. The incorrect equation was written as

$$\frac{q_i}{q_1} = \frac{1 - m_i s}{1 - ms} \tag{1}$$

In deriving this equation a compaction effect was neglected. In Figure 1 an increase in cake thickness from L to $L + dL$ in time $d\theta$ is shown. At the same time the thickness increases by dL, the original cake is compacted to thickness L'. The amount of cake laid down must equal dL'. Use of dL rather than dL' led to the error in the original derivation.

The mass of cake deposited is

$$dw = \rho_s (1 - \epsilon_i)\, dL' \tag{2}$$

Since $w = \rho_s(1 - \epsilon_{av})\, L$, dw is also given by

$$dw = \rho_s(1 - \epsilon_{av})\, dL - \rho_s L d\epsilon_{av} \tag{3}$$

Equating (2)and (3) one obtains

$$dL' = \frac{1 - \epsilon_{av}}{1 - \epsilon_i} dL - \frac{L}{1 - \epsilon_i}\, d\epsilon_{av} \tag{4}$$

The fraction of freshly deposited cake which flows out of dL into the cake is given by

$$\frac{dL' - dL}{dL'} = \frac{\epsilon_i - \epsilon_{av}}{1 - \epsilon_{av}} - \left[\frac{1 - \epsilon_i}{1 - \epsilon_{av}}\right] \left[\frac{L(d\epsilon_{av}/dL)}{1 - \epsilon_{av} - L(d\epsilon_{av}/dL)}\right] \tag{5}$$

If the average porosity is constant, the last term in (5) is zero.

The flow rate q_i at the surface layer can be obtained as follows

q_i = liquid flow rate in slurry — rate of liquid deposited in dL' + rate of liquid squeezed out of dL' (6)

It was previously demonstrated in the original derivation on a unit area basis that

$$\text{liquid flow rate in slurry} = \frac{1 - s}{\rho s} \frac{dw}{d\theta} \tag{7}$$

$$\text{rate of liquid deposited} = \frac{m_i - 1}{\rho} \frac{dw}{d\theta} \tag{8}$$

Equations (7) and (8) were Equations (9) and (10) in the original derivation. The liquid squeezed out of dL' and added to the internal flow rate is

$$\frac{dL' - dL}{dL'} \left(\frac{\epsilon_i}{1 - \epsilon_i}\right) \frac{1}{\rho_s} \frac{dw}{d\theta} \tag{9}$$

When ϵ_{av} is constant, this becomes

$$\left(\frac{\epsilon_i - \epsilon_{av}}{1 - \epsilon_{av}}\right) \left(\frac{\epsilon_i}{1 - \epsilon_i}\right) \frac{1}{\rho_s} \frac{dw}{d\theta} \tag{10}$$

then

$$q_i = \left[\frac{1 - s}{s} - (m_i - 1) + \frac{\rho}{\rho_s} \left(\frac{\epsilon_i - \epsilon_{av}}{1 - \epsilon_{av}}\right) \left(\frac{\epsilon_i}{1 - \epsilon_i}\right)\right] \frac{1}{\rho} \frac{dw}{d\theta} \tag{11}$$

or

$$q_i = \left[\frac{1 - m_i s}{s} + \frac{\rho}{\rho_s} \left(\frac{\epsilon_i - \epsilon_{av}}{1 - \epsilon_{av}}\right) \left(\frac{\epsilon_i}{1 - \epsilon_i}\right)\right] \frac{1}{\rho} \frac{dw}{d\theta} \tag{12}$$

From Equation (8) of the previous article the flow rate q_1 at the medium was found to be

$$q_1 = \left(\frac{1 - ms}{\rho s}\right) \frac{dw}{d\theta} - \frac{w}{\rho} \frac{dm}{d\theta} \tag{13}$$

If ϵ_{av} is constant, $dm/d\theta$ is zero. For this special case q_1 reduces to the first term on the right-hand side of (12). Dividing q_1 into q_i one obtains

$$\frac{q_i}{q_1} = \frac{1 - m_i s}{1 - ms} + \frac{\rho}{\rho_s} \frac{\epsilon_i - \epsilon_{av}}{1 - \epsilon_{av}} \frac{\epsilon_i}{1 - \epsilon_i} \frac{s}{1 - ms} \tag{14}$$

$$= \frac{1 - m_i s}{1 - ms} \left[1 + \frac{\epsilon_i - \epsilon_{av}}{1 - \epsilon_{av}} \frac{m_i - 1}{1 - m_i s} s\right] \tag{15}$$

When s approaches its limiting, maximum value $1/m_i$, q_i/q_1 approaches ϵ_i instead of zero as previously assumed.

In discussing a flow rate variation for the data of Grace (*1*) for zinc sulfide, q_1 was calculated to be 8.27, q_i obtained from Equation (1). Since $\epsilon_i = 0.94$, the variation was much smaller. Use of Equation (15) indicates that the flow ratio in this example was only about 1.05. The values in Table 1 of "The Role of Porosity in Filtration: IV" should be modified in accordance with (15). In general q_1 will not be over twice the value of q_i.

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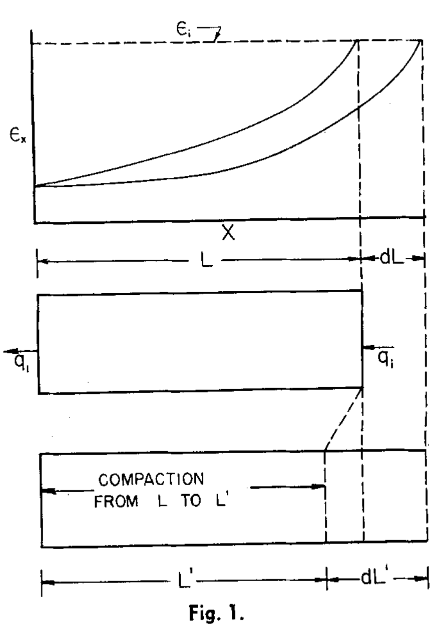

Fig. 1.